Galilei-invariant gauge symmetries in Fokker–Planck dynamics with logarithmic diffusion and drift terms

2007 J. Phys. A: Math. Theor. 40 13467

(http://iopscience.iop.org/1751-8121/40/44/023)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **40** (2007) 13467–13478 doi:10.1088/1751-8113/40/44/023

Galilei-invariant gauge symmetries in Fokker–Planck dynamics with logarithmic diffusion and drift terms

J A Cardeal[1], **M de Montigny**[2,3], **F C Khanna**[3,4], **T M Rocha Filho**[5] **and A E Santana**[5]

[1] Departamento de Ciências Exatas, Universidade Estadual de Feira de Santana, 44031-460, Feira de Santana, Bahia, Brazil
[2] Campus Saint-Jean, University of Alberta 8406-91 Street, Edmonton, Alberta T6C 4G9, Canada
[3] Theoretical Physics Institute, University of Alberta, Edmonton, Alberta T6G 2J1, Canada
[4] TRIUMF, 4004 Wesbrook Mall, Vancouver, British Columbia V6T 2A3, Canada
[5] Instituto de Física, Universidade de Brasília, CP 04455, CEP 70919-970, Brasília, DF, Brazil

Received 26 June 2007, in final form 19 September 2007
Published 16 October 2007
Online at stacks.iop.org/JPhysA/40/13467

Abstract

A tensorial approach to Galilean invariance is utilized, together with Lie symmetries of differential equations, in order to derive equations of Fokker–Planck type containing a logarithmic diffusion tensor and drift term. The formalism is based on the projection from an extended (by one space-like dimension) Minkowski manifold $\mathcal{G}$ to the usual Newtonian spacetime, so that non-relativistic models are described by manifestly covariant Lagrangians. In this paper, we obtain the Fokker–Planck equations from the Euler–Lagrange equations with the extended manifold $\mathcal{G}$ by using a specific choice of the gauge condition. We work in (1+1) spacetime and carry out the analysis for both Abelian and non-Abelian symmetries.

PACS numbers: 11.15.−q, 02.20.Sv, 05.20.Dd

1. Introduction

Lie groups and symmetries found their way into physics more than a century ago, and they are widely used in many fields. Introduced by Sophus Lie in his study of symmetry transformation groups of differential equations [1], they are the foundations of many powerful methods to classify and determine solutions of differential equations [2]. After the advent of quantum mechanics and quantum field theory, Lie groups became one of the most fruitful mathematical tools of modern theoretical physics. In his seminal work, Wigner provided the foundations for characterizing elementary particles from Lie group representations of the Poincaré group [3]. Along the same lines, non-relativistic quantum physics can be described from representations of the Galilei group [4].

In this work, we apply this approach to deduce Fokker–Planck transport equations which describe physical situations where such type of equations are either not known or cumbersome. The present approach can be useful, for instance, in the study of transport in QCD plasmas [5–7]. Fokker–Planck equations in (1+1) dimensions with an arbitrary drift and diffusion terms were fully classified by Rudra [8] and Cicogna and Vitali [9, 10]. The (2+1)-dimensional case with constant diffusion was examined by Finkel [11]. A non-constant diffusion term can be relevant for phenomena such as quantum chaos and nucleation in metals [14, 20–24]. Also, it is reasonable to expect that systems with confinement, such as a QCD plasma, can be described by transport equations with drift and diffusion terms that go to zero, or diverge, in the confinement realm.

Lie symmetries have often been invoked to solve and generalize the Fokker–Planck equation with non-trivial drift and diffusion terms [12–19]. The central ingredient of these approaches is a basic starting symmetry which is usually considered as the symmetry of a more restrictive set of equations. For instance, the well-known symmetry group of the diffusion equation has been used to derive Fokker–Planck equations [16], but the choice of the initial symmetry and its realizations is arbitrary. One possibility is to exploit symmetries of the underlying stochastic equations [25, 26], which should be related to the Fokker–Planck equation.

In this paper, we use different realizations of three-dimensional Lie algebras in (1+1) spacetime. In addition to some known results, we obtain a Fokker–Planck equation with logarithmic drift and diffusion coefficients. Recently, computer simulations have stimulated interest in this class of Fokker–Planck equations in relation with quantum chaos and experimental results describing nucleation in metals [14, 20–24]. Let us emphasize that, in a broad sense, our results show that a Fokker–Planck dynamics can be thought of as a field theory fully defined in terms of symmetries. Our starting point is based on a Galilean-covariant metric approach which describes the non-relativistic theories by working in a Minkowski spacetime $\mathcal{G}$ extended by one spatial dimension, with light-cone coordinates [27–29]. In the present paper, we gather and expand the results in [30, 31], emphasizing that the drift and diffusion tensors can be explicitly obtained using a systematic scheme based on realizations of Lie algebras.

The plan of this paper is as follows. In section 2, we briefly review the derivation of classes of Fokker–Planck equations with $U(1)$ gauge-invariant Lagrangians. In section 3, we present a prescription to enforce a specific symmetry with equations of Fokker–Planck type on a (1+1) spacetime with $SL(2, R)$ symmetry. These results are then extended to a non-Abelian $SU(2)$ gauge theory in section 4. Consideration of this example is intended to get some understanding as to how to proceed with a non-Abelian gauge theory. Concluding remarks are presented in section 5. We argue that the study of the $SL(2, R)$ symmetry is relevant for quantum chaos and nucleation in metals. The example of non-Abelian gauge theory is presented as a simple illustration of a transport equation borrowing elements from a color theory.

2. Summary of Abelian Fokker–Planck Lagrangians

In order to introduce the notion of Galilei covariance [30] we consider the five-vector $p_\mu = (\mathbf{p}, p_4, p_5)$, where $\mathbf{p}$ is the Euclidean momentum vector, $p_5 = -H/v$ (with H the energy, and v has the units of velocity) and $p_4 = -mv$ (with mass m). Then, with the metric

$$\eta = \delta_{ij}\,\mathrm{d}x^j \otimes \mathrm{d}x^i - \mathrm{d}x^4 \otimes \mathrm{d}x^5 - \mathrm{d}x^5 \otimes \mathrm{d}x^4, \tag{1}$$

we find the following dispersion relation:

$$p_\mu p^\mu = p_\mu p_\nu g^{\mu\nu} = \mathbf{p}^2 - 2p^4 p^5 = k^2, \tag{2}$$

where k is a constant. This relation is consistent with the fact that a free particle with mass m has total energy equal to $H = \mathbf{p}^2/2m$, so that

$$p_\mu p^\mu = \mathbf{p}^2 - 2mH = 0.$$

Henceforth, we shall take k equal to zero, or absorb it within H.

The metric in equation (1) defines a (4+1) Minkowski space with a 15-dimensional Poincaré algebra (corresponding to the inhomogeneous group of linear transformations in the extended configuration space), which contains both the usual ten-dimensional Poincaré algebra and the Galilei algebra. This extended manifold provides a unifying scheme for treating both the relativistic and non-relativistic physics in (3+1) dimensions.

The five-vector p^μ can be related to the canonical set of five conjugate coordinates $q^\mu = (\mathbf{q}, q^4, q^5)$ in a configuration space $\mathcal{G}$ with metric η. In the present approach, these variables are interpreted as follows: $\mathbf{q}$ are the canonical coordinates conjugate to $\mathbf{p}$; q^4 is conjugate to $p^4 = H/v$, so that q^4 is a time coordinate; q^5 is conjugate to $p^5 = mv$, the mass m up to a redefinition on the unit of mass. The mass q^5 can then be written as a function of $\mathbf{q}$ and q^4 which obeys an expression for the analogue of an interval in $\mathcal{G}$:

$$q_\mu q^\mu = q^\mu q^\nu g_{\mu\nu} = \mathbf{q}^2 - 2q^4 q^5 = s^2.$$

Since such an expression is the canonical-coordinate counterpart of the dispersion relation, equation (2), we choose $s = 0$, which corresponds to $k = 0$. This leads to

$$q^5 = \mathbf{q}^2/2q^4.$$

With $q^4 = vt$, it follows that $q^5 = \mathbf{q}^2/2vt$. In short, we have defined an embedding of the Euclidean space into $\mathcal{G}$:

$$(\mathbf{q}, t) \to q^\mu = (\mathbf{q}, q^4, q^5).$$

Let us consider $U(1)$ gauge-invariant Lagrangian written in terms of the 2-form tensor field F, with components $F^{\mu\nu}$:

$$\mathcal{L} = -\tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}. \tag{3}$$

The tensor $F^{\mu\nu}$ is written in terms of the Abelian gauge fields J as

$$F_{\mu\nu} = \partial_\mu J_\nu - \partial_\nu J_\mu,$$

where J remains to be specified. This leads to the usual Euler–Lagrange equations:

$$\partial^\mu \partial_\mu J^\nu - \partial^\nu \partial_\mu J^\mu = 0. \tag{4}$$

The Lagrangian $\mathcal{L}$ is invariant under the gauge transformation

$$J^\mu \to \bar{J}^\mu = J^\mu + \partial^\mu h(x).$$

Taking the gauge condition as being

$$\partial^\mu \partial_\mu J^\nu = 0,$$

the function $h(x)$ satisfies the constraint equation

$$\partial^\mu \partial_\mu h(x) = \beta,$$

where β is an arbitrary constant, and also

$$\partial_\mu J^\mu = \alpha,$$

where α is another arbitrary constant, which we can take equal to zero. Then the Euler–Lagrange equation is written as

$$\partial_\mu J^\mu = 0. \tag{5}$$

In order to specify the five-dimensional vector field theory, we define this gauge theory on a pseudo-Riemannian manifold $\mathcal{R}(\mathcal{G})$, with $g^{\mu\nu}(x)$ being the components of the metric tensor [30, 31]:

$$g = P(x)B_{ij}(x)\,\mathrm{d}x^j \otimes \mathrm{d}x^i - \mathrm{d}x^4 \otimes \mathrm{d}x^5 - \mathrm{d}x^5 \otimes \mathrm{d}x^4, \tag{6}$$

such that, at each point of $\mathcal{R}(\mathcal{G})$, the space is the locally flat $\mathcal{G}$. The result is to write the components of J^μ as

$$J^i = A^i(x)P(x) + \partial_j P(x)B^{ij}(x),$$
$$J^4 = P(x),$$
$$J^5 = 0.$$

We obtain from equation (5) that

$$\partial_t P(\mathbf{x}, t) = \frac{\partial}{\partial x^i}\left[-A^i(\mathbf{x}, t)P(\mathbf{x}, t) + \frac{\partial}{\partial x^j}B^{ij}(\mathbf{x}, t)P(\mathbf{x}, t)\right], \tag{7}$$

which is the Fokker–Planck equation with the *drift term* $A^i(\mathbf{x}, t)$, and the *diffusion tensor* $B^{ij}(\mathbf{x}, t)$. We can take $P(\mathbf{x}, t)$ to be a real positive and normalized function, so that it can be interpreted as a (covariant) probability density.

In the following section, we address the problem of obtaining explicitly in a (1+1)-dimensional spacetime the diffusion and drift terms by considering general symmetry arguments.

3. Symmetries of drift terms and diffusion tensors

3.1. Procedure to enforce symmetries

There are various types of symmetries considered hereafter. First, there is the Galilean spacetime symmetry, implemented here with the extended manifold. Second, the Lie symmetries of differential equations, that is, the transformations (in general, of both the dependent and independent variables) which leave these equations unchanged. And third, we exploit gauge-symmetric Lagrangians.

In order to obtain explicit expressions for the drift vector and diffusion tensor terms in equation (7), we proceed with a group theoretical approach based on symmetries of the differential equations [2, 19]. In order to do so, we take a generic element G of the symmetry Lie group. If G is connected to the identity, we have

$$G = \exp\left(\sum_{k=1}^{m} \alpha_k T_k\right), \tag{8}$$

where T_k denotes the infinitesimal symmetry generators, and the coordinates α_k are finite numbers. A linear partial differential equation can be cast into the following general form:

$$\Delta(x)\theta(x) = 0, \tag{9}$$

where $\Delta(x)$ is a partial differential (field) operator defined in $\mathbf{R}^m$ with coordinates $x = (x_1, x_2, \ldots, x_m)$, and $\theta(x)$ is a function of $\mathbf{R}^m$. As explained previously [2, 12, 13, 19, 32], to say that G is a symmetry group of equation (9) means that for a symmetry transformation generator $L(x)$ which belongs to the Lie algebra of G, we have

$$L(x)\Delta(x)\theta(x) = 0.$$

Since we can write this generator in terms of the generators T_k as $L(x) = a_k T_k(x)$, we can rewrite the invariance condition above as follows:

$$[T_k(x), \Delta(x)] = r_k(x)\Delta(x), \qquad k = 1, \ldots, \dim(G), \tag{10}$$

where $r_k(x)$'s are functions in $\mathbf{R}^m$.

Our purpose is to use equation (10) with Δ, a Fokker–Planck type differential operator, and T_k the generators of a given Lie algebra. In order to understand how this approach works, we start by considering the low dimensional algebras. Three-dimensional algebras are classified [33] and can be used to determine the diffusion and drift terms, as shown in this paper. The choice of a given algebra from the underlying physics is still an open question. The present work is an initial effort in this direction by a systematic study of all representations of lower dimensional algebras. To show how this is done, we consider the algebra with the commutation relations

$$[T_1, T_2] = 2T_2, \qquad [T_3, T_1] = 2T_3, \qquad [T_2, T_3] = T_1. \tag{11}$$

This algebra is isomorphic to the $sl(2, \mathbf{R})$ algebra, which has physical applications in the case of quantum chaos and nucleation in metals. Clearly, it is possible to define many realizations of this Lie algebra in terms of vector fields, even for a specific number of manifold dimensions. All computations with symmetries were performed using the package *SADE* developed by the mathematical physics group at the University of Brasília [34], written in the symbolic manipulation system MAPLE.

We consider here realizations of this algebra in (1+1) spacetime of the form:

$$\begin{aligned} T_1 &= \partial_t, \\ T_2 &= k_1(x,t)\partial_t + k_2(x,t)\partial_x + h_1(x,t), \\ T_3 &= k_3(x,t)\partial_t + k_4(x,t)\partial_x + h_2(x,t), \end{aligned} \tag{12}$$

where $k_1(x,t), k_2(x,t), k_3(x,t)$ and $k_4(x,t)$ are functions of x and t constrained by the commutation relations in equation (11). If we substitute the expressions for T_1 and T_2 from equation (12) with $h_1 = h_2 = 0$ into the commutator $[T_1, T_2]$ of (11), then we find

$$[\partial_t k_1(x,t) - 2k_1(x,t)]\partial_t + [\partial_t k_2(x,t) - 2k_2(x,t)]\partial_x = 0$$

which leads to $\partial_t k_1(x,t) = 2k_1(x,t)$ and $\partial_t k_2(x,t) = 2k_2(x,t)$, the solutions of which are

$$k_1(x,t) = f_1(x)\exp(2t), \qquad k_2(x,t) = f_2(x)\exp(2t). \tag{13}$$

Similarly, with the commutator of T_1 and T_3 in equation (11), the realization of equation (12) gives us

$$[\partial_t k_3(x,t) + 2k_3(x,t)]\partial_t + [\partial_t k_4(x,t) + 2k_4(x,t)]\partial_x = 0,$$

and thence

$$k_3(x,t) = f_3(x)\exp(-2t), \qquad k_4(x,t) = f_4(x)\exp(-2t). \tag{14}$$

Finally, by using the third commutator, $[T_2, T_3]$, of equation (11), together with equation (12), we obtain

$$k_1(x,t)\partial_t k_3(x,t) - k_3(x,t)\partial t k_1(x,t) - k_4(x,t)\partial_x k_1(x,t) + k_2(x,t)\partial_x k_3(x,t) = 1 \tag{15}$$

and

$$k_1(x,t)\partial_t k_4(x,t) - k_3(x,t)\partial_t k_2(x,t) + k_2(x,t)\partial_x k_4(x,t) - k_4(x,t)\partial_x k_2(x,t) = 0. \tag{16}$$

From equations (13) and (14), we find that equations (15) and (16) lead to

$$\begin{aligned} &f_2(x)\partial_x f_3(x) - 4f_1(x)f_3(x) - f_4(x)\partial_x f_1(x) = 1, \\ &f_2(x)\partial_x f_4(x) - 2f_1(x)f_4(x) - 2f_3(x)f_2(x) - f_4(x)\partial_x f_2(x) = 0. \end{aligned} \tag{17}$$

Therefore, we may summarize by rewriting equation (12) as

$$\begin{aligned} T_1 &= \partial_t, \\ T_2 &= f_1(x)\exp(2t)\partial_t + f_2(x)\exp(2t)\partial_x, \\ T_3 &= f_3(x)\exp(-2t)\partial_t + f_4(x)\exp(-2t)\partial_x, \end{aligned} \tag{18}$$

where the f's satisfy equation (17).

In the following section, we explore solutions of such equations and use the resulting realizations in equation (10). This procedure allows us to specify the form of the drift and diffusion terms in Fokker–Planck equations.

3.2. *Examples of Fokker–Planck operators*

We consider a Fokker–Planck equation in a (1+1)-dimensional spacetime:

$$\Delta(x,t)P(x,t) = \partial_t P(x,t) + \partial_x[A(x,t)P(x,t)] + \partial_{xx}[-B(x,t)P(x,t)] = 0. \tag{19}$$

Let us now look for a realization of equations (17) with which we can consider as a candidate to describe a system confined in some region of space. (In such a situation, the variable x in the symmetry generators stands for the space coordinate.) As discussed in the introduction, one possibility for a limited range, say $[x_a, x_b] \in R$, is to consider drift and diffusion terms vanishing at $x = x_a$ and $x = x_b$ (another possibility is a divergent drift, but we will not consider it here). But in our approach, this kind of behavior is determined by the nature of the symmetry generators, in particular, those associated with space transformations. Then we have to specify $f_i(x), i = 1, 2, 3, 4$, in equations (18) with the physical characteristic of the type of confinement considered here, which can be described by functions of the form $f(x) \sim x^n \ln x$ (see figure 1). Then we find that the solutions of equation (17) are

$$\begin{aligned} &f_1(x) = c_1, && f_3(x) = -\frac{1}{4c_1}, \\ &f_2(X) = c_2 x^n \ln x, && f_4(x) = \frac{c_2}{4c_1^3} x^n \ln x. \end{aligned} \tag{20}$$

Then the generators in equations (18) are given by

$$\begin{aligned} T_1 &= \partial_t, \\ T_2 &= \exp(2t)\partial_t + \tfrac{1}{2}x^n \ln x \exp(2t)\partial_x, \\ T_3 &= -\exp(-2t)\partial_t + \tfrac{1}{2}x^n \ln x \exp(-2t)\partial_x. \end{aligned} \tag{21}$$

These operators fulfil the commutation relation in equation (11). A more general expression of such operators, including a phase symmetry is given, for $n = 1$, by

$$T_1 = \frac{1}{\gamma}\partial_t + \frac{\delta}{2\epsilon} + \frac{1}{2}, \tag{22}$$

$$T_2 = \frac{1}{2}\,\mathrm{e}^{2\gamma t}\left(2x\ln x\,\partial_x + \frac{1}{\gamma}\partial_t + 1\right), \tag{23}$$

$$T_3 = \frac{1}{2}\,\mathrm{e}^{-2\gamma t}\left(2x\ln x\,\partial_x - \frac{1}{\gamma}\partial_t + \frac{\gamma}{\varepsilon}\ln x - \frac{\delta}{\epsilon}\right). \tag{24}$$

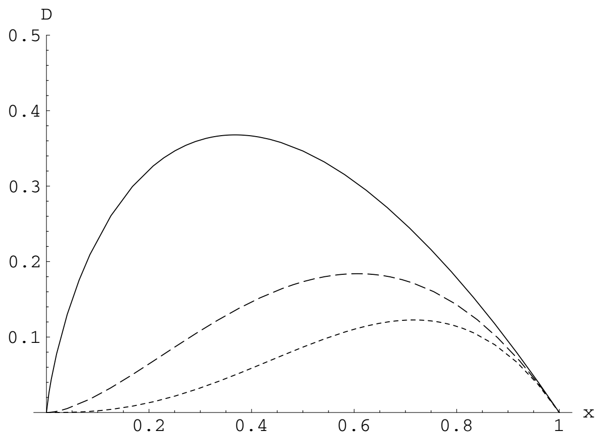

Figure 1. Behavior of functions $x^n \ln x$, for $n = 1, 2, 3$.

Using these results in equations (10), with the general expression for $\Delta(x)$ given in equations (19), we obtain

$$\Delta(x,t) = \partial_t + \gamma - \frac{C}{\ln x} + (2\varepsilon x + 4\varepsilon x \ln x + 2\gamma x \ln x - 2\delta x)\partial_x + 4\varepsilon x^2 \ln x\, \partial_{xx}. \tag{25}$$

This is a Fokker–Planck equation where the drift and the diffusion terms are given, respectively, by

$$A(x) = \gamma x \ln x - 2\delta x, \tag{26}$$

$$B(x) = 4\varepsilon x^2 \ln x. \tag{27}$$

This result for the drift term was first analyzed by Lehnik [20], to describe nucleation in metals. More recently, it has been pointed out by Dettman and Cohen [22] that due to quantum chaos effects, some diffusion processes would be described by a logarithmic diffusion term. For nonlinear Fokker–Planck equation including such log terms, see [14].

4. Non-Abelian logarithmic-like drift and diffusion terms

Consider a gauge-invariant non-Abelian Lagrangian defined on the manifold $\mathcal{G}$ discussed in section 2:

$$\mathcal{L} = -\tfrac{1}{4} F^{a\mu\nu} F_{a\mu\nu}, \tag{28}$$

where the Latin index a stands for the gauge group, with generators t^a, $a = 1, \ldots, n$, satisfying the Lie algebra $[t^a, t^b] = C_c^{ab} t^c$, where C_c^{ab} are structure constants of the gauge group (summation convention over Latin indices is assumed). The field strength tensor $F^a_{\mu\nu}$ is given by

$$F_{\mu\nu a} = \partial_\mu J_{\nu a} - \partial_\nu J_{\mu a} - \lambda C_a^{bc} J_{\mu b} J_{\nu c},$$

for which the equation of motion is written as $D_a^{\mu b} F_{\mu\nu b} = 0$, where $D_a^{b\mu}$ is the covariant derivative $D_a^{\mu b} = \partial^\mu \delta_a^b + \lambda C_a^{bc} J_c^\mu$. Using the gauge condition $\partial_\mu \partial^\mu J_{\nu a} = 0$, the equations of motion for each component of J are

$$\partial_\nu \partial_\mu J_a^\mu = \lambda C_a^{bc} \partial_\mu \left(J_c^\mu J_{\nu b} \right) + \lambda C_a^{bc} J_c^\mu \partial_\mu J_{\nu b} + \lambda C_a^{cb} J_c^\mu \partial_\nu J_{\mu b} + \lambda^2 C_a^{cb} C_b^{de} J_c^\mu J_{\mu d} J_{\nu e}. \tag{29}$$

Despite the nonlinear structure of these equations, a Fokker–Planck system of equations can be recognized if J is defined as in the Abelian case, and if we discard the nonlinear terms in equation (29) [30], such that

$$\partial_\mu \partial_\nu J_a^\mu = 0.$$

As a consequence

$$\partial_\mu J_a^\mu = \alpha, \tag{30}$$

where α is a constant. If we choose $\alpha = 0$, we obtain equation (5), which leads to a Fokker–Planck equation for each gauge index a.

On the other hand, by considering $\alpha \ll 1$, then equation (29) reduces, up to second-order terms in $\lambda\alpha$, to

$$\partial_\nu \left(\partial_\mu J_a^\mu + \lambda C_a^{bc} J_c^\mu J_{\mu b} \right) = 2\lambda C_a^{bc} J_c^\mu \partial_\mu J_{\nu b} + \lambda C_a^{bc} \left(\partial_\nu J_c^\mu \right) J_{\mu b}. \tag{31}$$

The left-hand side of this equation can be integrated for each $\nu = 1, \ldots, 5$, such that the right-hand side results in a nonlocal term along each direction. In a heuristic construction, if we discard, as a first approximation, these nonlocal terms we obtain the following nonlinear [30] equation:

$$\partial_\mu J_a^\mu + \lambda C_a^{bc} J_c^\mu J_{\mu b} = C, \tag{32}$$

where C is a constant. In the following we chose $C = 0$ for illustrative purposes.

Let us consider as an example the $SU(2)$ symmetry with J_a^μ defined by

$$J_a^i = \epsilon_{aij} \left[A_j^k P_k + \partial_k \left(B_j^{nk} P_n \right) \right],$$
$$J_a^4 = P_a,$$
$$J_a^5 = 0,$$

where both gauge and tensor indices are of the same nature (that is, i, j, k and a, b, c are all equal to $1, 2, 3$), $A_j^k = A_j^k(x)$ describes the drift term (which is now a rank-two tensor (taking into account the vector and the gauge index), while $D_j^{nk}(x)$ stands for the diffusion term. Note that this definition can be developed along the reasoning used in the Abelian case. From equation (32), it follows that $\epsilon_{abc} J_c^\mu J_{\mu b} = \epsilon_{abc} J_{ic} J_{ib} = 0$. Hence

$$\partial_t P_a = \epsilon_{aji} \left[\partial_i \left(A_j^b P_b \right) + B_j^{cb} \partial_i \partial_b P_c \right]. \tag{33}$$

Let us analyze the content of this Fokker–Planck-like equation in some particular situations. First, define

$$P_2 = P_3 = P,$$
$$A_2^1 = f(z), \qquad A_3^1 = g(y),$$
$$B_2^{13} = a(y), \qquad B_3^{12} = b(z),$$

where P is a constant and the other components of A_j^b and B_j^{cb} are zero. With the above expressions for the drift terms A_2^1 and A_3^1, and the diffusion tensor components $B_2^{13} = B_3^{12}$, we are assured that we have an arbitrary process for this theory with color as the gauge index, and yet, with the characteristics of a Fokker–Planck-like dynamics. Indeed, if we write

$$P_1(y, z, t) = \varphi(y)\phi(z)\,\mathrm{e}^{wt},$$

we get

$$w = \frac{1}{\phi(z)}\left[\frac{d^2}{dz^2}[b(z)\phi(z)] + \frac{d}{dz}[f(z)\phi(z)]\right] - \frac{1}{\varphi(y)}\left[\frac{d^2}{dy^2}[a(y)\varphi(y)] + \frac{d}{dy}[g(y)\varphi(y)]\right].$$

Therefore, with

$$\frac{1}{\phi(z)}\left[\frac{d^2}{dz^2}[b(z)\phi(z)] + \frac{d}{dz}[f(z)\phi(z)]\right] = F_1, \tag{34}$$

$$\frac{1}{\varphi(y)}\left[\frac{d^2}{dy^2}[a(y)\varphi(y)] + \frac{d}{dy}[g(y)\varphi(y)]\right] = F_2, \tag{35}$$

we have $w = F_1 - F_2$. By multiplying equation (34) by $\exp(F_1 t)$, we use

$$F_1\phi(z,t) = \frac{\partial}{\partial t}\phi(z,t),$$

together with

$$\phi(z,t) = \phi(z)\exp(F_1 t),$$

to find the equation

$$\frac{\partial}{\partial t}\phi(z,t) = \frac{\partial^2}{\partial z^2}[b(z)\phi(z,t)] + \frac{\partial}{\partial z}[f(z)\phi(z,t)]. \tag{36}$$

Similarly, equation (35) becomes

$$\frac{\partial}{\partial t}\varphi(y,t) = \frac{\partial^2}{\partial y^2}[a(y)\varphi(y,t)] + \frac{\partial}{\partial y}[g(y)\varphi(y,t)]. \tag{37}$$

Note that equations (36) and (37) are as general as equation (19); hence we can use the procedure developed in the last two sections, based on the algebra given by equation (11), to provide expressions for $a(y), g(y), b(z)$ and $f(z)$. Let us consider the possibility of logarithmic terms in these functions such as

$$a(y) = 4\varepsilon y^2 \ln y, \qquad g(y) = 2\delta y + \gamma y \ln y,$$
$$b(z) = 4\varepsilon z^2 \ln z, \qquad f(z) = 2\delta z + \gamma z \ln z.$$

For the steady state, we have to solve the following equation:

$$\Phi'(x) + F(x)\Phi + c_1(x) = 0,$$

where $\Phi(x)$ stands for $\varphi(y)$ and $\phi(z)$, while $c_1(x)$ is given by $c/b(x)$, and

$$F(x) = [(4\varepsilon + 2\delta)x + (8\varepsilon + \gamma)x \ln x]/b(x).$$

A solution for $c_1 = 0$ is given by

$$\Phi(x) = c_2 \exp[-G(x)],$$

where

$$G(x) = (8\varepsilon + \gamma)\frac{1}{4\varepsilon}\ln x + \frac{1}{4\varepsilon}(4\varepsilon + 2\delta)\ln(\ln x).$$

The solution, $P_1(y,z)$, is then given by

$$P_1(y,z) = \varphi(y)\phi(z) = \exp[G(y) + G(z)],$$

where A is a constant.

Recent experiments at the relativistic heavy ion collider (RHIC) have revealed that the deconfined quark–gluon matter thereby obtained appears to behave like a strongly interacting quark–gluon plasma fluid [35, 36]. Such fluids seem to have a very low viscosity, so that it flows approximately like a perfect fluid. However, a detailed study would require using a transport equation, with consideration to the non-Abelian nature of quarks and gluons in order to deduce effectively the properties of the plasma. In this context, many studies of Yang–Mills fluid dynamics have been performed by Jackiw and co-workers, taking into account the relativistic as well as non-relativistic models [37]. Along the same directions, we have inferred a Fokker–Planck equation via a variational principle with gauge-invariant Lagrangians [30, 31]. A full consideration of the quark–gluon plasma is necessary to consider the symmetry group of the quarks plus the $SU(3)$ gauge group. This will likely require a coupled set of Fokker–Planck equations. Such a study is being undertaken and will be presented later on. However, the present work already indicates the role of symmetry in obtaining the structure of drift and diffusion terms in the Fokker–Planck equation.

We have thus examined $U(1)$ gauge-invariant Lagrangians and the generalization to non-Abelian theory, considering, as an example, the $SU(2)$ symmetry for color-like Ornstein–Uhlenbeck processes [30, 31]. This type of gauge-invariant formulation can be useful for various different problems. For instance, for the gauge group, the drift and diffusion terms were derived with the physical content of a metric tensor on a pseudo-Riemannian manifold defined in such a way that $\mathcal{G}$ is the local manifold.

5. Concluding remarks

In this paper, we have constructed equations of Fokker–Planck type by enforcing various symmetries: (i) Galilean invariance is implemented in an extended Minkowski spacetime to show that the usual Fokker–Planck equation presents $U(1)$ symmetry; (ii) the theory of Lie symmetries of differential equations is used to obtain an explicit expression for the drift and diffusion terms; (iii) gauge symmetry is implemented by using the Galilean covariance, resulting in a Fokker–Planck Lagrangian including a non-Abelian gauge color index. Items (i) and (ii) were discussed to some extent earlier [30]. However, here we have focused our attention on finding an explicit form for the drift and diffusion terms. In particular, we obtained for the first time these coefficients in the log-functional form by using general physical motivations supported by symmetries: we have demanded that the drift and the diffusion coefficients be written in a way consistent with a system confined to a region of space. This provides a guide as to introducing the symmetry generators with log-dependence, resulting in a log-dependent Fokker–Planck operator. However, let us observe that the Fokker–Planck equation constructed from this operator presents, as it would be expected, symmetries other than the original group. In our case, it is evident that the Fokker–Planck equation is invariant under dilation, for instance. Yet this symmetry is not described by the algebra we have utilized. This is a consequence of considering linear equations.

Such an observation points to the open problem of selecting the symmetry to obtain classes of Fokker–Planck equations. Up to now there is no specific criteria for that. But we can proceed further by analyzing the classification of Lie algebras, starting from algebras of the lowest dimensions. This classification is available in the mathematics literature [33]. It is not the case for their realizations which are of most interest to physicists. It is our contention that by following the scheme presented here, it is possible to carry out such an analysis. A study which extends such ideas is currently in progress and will be presented in a separate publication.

Finally, it is worth noting that log terms in the Fokker–Planck equation have theoretical and experimental interests, since they are associated with the problem of nucleation [20] and quantum chaos [22]. In addition, we have generalized this log-dependent Fokker–Planck equation to include non-Abelian gauge indices. It is important to emphasize that the present approach may be extended to non-Abelian gauge fields with $SU(3)$ symmetry that has direct applications to quark–gluon plasma, as observed in experiments at RHIC with colliding heavy ion beams. This is a subject that needs to be developed further and will be presented in a separate paper.

Acknowledgments

The authors would like to thank Jorge Malbouisson for his interest in this paper, and the granting agencies CNPq (Brazil) and NSERC (Canada) for partial financial support.

References

[1] Lie S 1881 *Arch. Math.* **6** 328
Ibragimov N H and Lie S (transl.) 1994 On integration of a class of linear partial differential equations by means of definite integrals vol 2 *CRC Handbook of Lie Group Analysis of Differential Equations* (New York: CRC Press) pp 473–508
[2] Olver P J 1993 *Applications of Lie Groups to Differential Equations* (Berlin: Springer)
[3] Wigner E P 1939 *Ann. Math.* **40** 149
[4] Lévy-Leblond J-M 1967 *Commun. Math. Phys.* **6** 286
[5] Selikhov A V and Gyulassy M 1994 *Phys. Rev.* C **49** 49
[6] Blaizot J-P and Iancu E 1999 *Nucl. Phys.* **557** 183
[7] Litim D F and Manuel C 2002 *Phys. Rep.* **364** 451
[8] Rudra P 1990 *J. Phys. A: Math. Gen.* **23** 1663
[9] Cicogna G and Vitali D 1989 *J. Phys. A: Math. Gen.* **22** L453
[10] Cicogna G and Vitali D 1990 *J. Phys. A: Math. Gen.* **23** L85
[11] Finkel F 1999 *J. Phys. A: Math. Gen.* **32** 2671
[12] Suzuki M 1983 *Physica* A **117** 103
[13] An I, Chen S and Guo H-Y 1984 *Physica* A **128** 520
[14] Silva E M, Rocha-Filho T M and Santana A E 2006 *J. Phys. Conf. Ser.* **40** 150
[15] Shtelen W M and Stogny V I 1989 *J. Phys. A: Math. Gen.* **22** L539
[16] Rudra P 1990 *J. Phys. A: Math. Gen.* **23** 1663
[17] Spichak S and Stognii V 1999 *J. Phys. A: Math. Gen.* **32** 8341
[18] Cherkasenko V 1995 *Nonlinear Math. Phys.* **2** 416
[19] Cardeal J A, Filho T M Rocha and Santana A E 2002 *Physica* A **308** 292
[20] Lehnik S H 1989 *J. Math. Phys.* **30** 953
[21] Jeffrey D J and Onishi Y 1984 *J. Fluid Mech.* **139** 261
[22] Dettman C P and Cohen E G D 2000 *J. Stat. Phys.* **101** 775
[23] Lo C F 2003 *Phys. Lett.* A **319** 110
[24] Pesz K 2002 *J. Phys. A: Math. Gen.* **35** 1827–32
[25] Misawa T 1994 *J. Phys. A: Math. Gen.* **27** L777
[26] Gaeta G and Quintero N R 1999 **32** 8485
Gaeta G 2000 **33** 4883
[27] Duval C, Burdet G, Künzle H P and Perrin M 1985 *Phys. Rev.* D **31** 1841
[28] Takahashi Y 1988 *Fortschr. Phys.* **36** 63
Takahashi Y 1988 *Fortschr. Phys.* **36** 83
[29] de Montigny M, Khanna F C and Santana A E 2001 *J. Phys. A: Math. Gen.* **34** 10921
[30] de Montigny M, Khanna F C and Santana A E 2003 *Physica* A **323** 327
[31] Santana A E, Cardeal J A, Khanna F C, de Montigny M and Rocha-Filho T M 2006 *PoS* (IC2006) p 012
[32] Fushchich W I and Nikitin A G 1994 *Symmetries of Equations of Quantum Mechanics* (New York: Allerton Press)
[33] Zhadanov R I and Ilano V I 1999 *J. Phys. A: Math. Gen.* **32** 7405

[34] Rocha-Filho T M and Figueiredo A 2007 [SADE] A Maple package for the symmetry analysis of differential equations *Computer Physics Communications* at press
[35] Adcox K *et al* (PHENIX Collaboration) 2005 *Nucl. Phys.* A **757** 184–283
[36] Shuryak E 2005 *Preprint* hep-th/0510123
Kolb P F x and Heinz U W 2003 *Preprint* nucl-th/0305084
[37] Jackiw R 2002 *Lectures on Fluid Dynamics: A Particle Theorist's View of Supersymmetric, Non-Abelian Fluid Mechanics and d-Branes* (Berlin: Springer)
Jackiw R, Li H, Nair V P and Pi S Y 2000 *Phys. Rev.* D **62** 085018
Bistrovic B, Jackiw R, Li H, Nair V P and Pi S Y 2003 *Phys. Rev.* D **67** 025013
Jackiw R 2004 *Preprint* hep-th/0410284
Bambah B A, Mahajan S M and Mukku C 2006 *Phys. Rev. Lett.* **97** 072301